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A THEORETICAL INVESTIGATION OF STRESSES NEAR THE CORNER OF AN ORTHOTROPIC ELASTIC ORTHOGONAL WEDGE

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Abstract—This theoretical investigation is concerned with stress distribution in the vicinity of the vertex of an orthotropic elastic orthogonal wedge subjected to admissible boundary conditions. By admissible boundary conditions it is meant that the normal and tangential boundary conditions $N(s)$ and $T(s)$, respectively, are such that $\int_0^\infty |N(s)|^2\,ds$ and $\int_0^\infty |T(s)|^2\,ds$ are finite. The axes of the orthotropy of the wedge are assumed to coincide with axes of the coordinate system. In this analysis, Fourier–Plancherel integral transform is used to solve the boundary value problems of orthotropic elastic half plane and to solve a system of integral equations for which the kernels $k_i(t, s)$, $i = 1, 2$, do not satisfy the necessary Fredholm's alternative

$$\int_0^\infty \int_0^\infty |k_i(t, s)|^2\,dt\,ds < \infty, \quad i = 1, 2.$$

The problem of elastic orthotropic orthogonal wedge is divided into four basic problems each of which is characterized by relevant generalized Green's functions. These generalized Green's functions are evaluated analytically as well as numerically. Knowing these generalized Green's functions, special formula is developed to calculate the stresses at any given point in the wedge for any arbitrary admissible boundary conditions.

A special form of Filon's method is used to evaluate improper integrals with rapidly oscillating integrands. The whole procedure of calculating stresses is illustrated by an example.

1. INTRODUCTION

Generally, the stress problems of infinite isotropic wedges, loaded in their plane are solved with the aid of polar coordinates using Mellin transform. Tranter[1] and recently Bogy[2] have analysed isotropic wedges using Mellin transform. On the other hand, Hetenyi[3] showed that for an isotropic quarter plane, the stresses distribution can be obtained by repeated superposition of known solutions of elastic half plane. He developed an algorithm to obtain the solution. Although these procedures gave satisfactory results to the problem, they cannot be easily extended for further problems of wedges without tedious work. For example, when the material is anisotropic and when the body forces and dynamic forces have to be taken into account these methods become extremely difficult to handle. Moreover, there is no simple way to get results of other problems of wedges without repeating the entire procedure. So, in this paper, it is shown that a more general problem of orthotropic wedge can be simply analysed by using Fourier–Plancherel integral transform and obtaining a set of simultaneous integral equations, the solution of which is obtained analytically. And this method of solution is much more general than other methods mentioned previously.

In Sections 2 and 3 a two dimensional boundary value problem for the elastic orthotropic orthogonal wedge is formulated. Fourier–Plancherel integral transform is used to obtain a stress function for an elastic orthotropic half plane. Also, the normal and tangential stresses at any point in the half plane is calculated. In Section 3 the problem of the wedge is divided into four basic problems and a detailed derivation of the method of solution is presented. The generalized Green's functions for each of the above-mentioned problems are obtained. It is also shown that this basic solution can simply be extended to other problems of the wedge by simple integration. Finally, the numerical results are shown.

2. FORMULATION AND SOLUTION OF BOUNDARY VALUE PROBLEM FOR ORTHOTROPIC HALF PLANE

The solution of the boundary value problem for an orthotropic half plane will be a suitable and convenient means of investigating the stress and strain distribution in an orthotropic elastic orthogonal wedge. Therefore, it is necessary to give a formulation and a brief abstract of the solution of boundary value problem for the half plane.

It is shown in[4] that the stress function $w(x, y)$ satisfies the following equation,

$$a_{22}\frac{\partial^4 w}{\partial x^4}-2a_{26}\frac{\partial^4 w}{\partial x^3\partial y}+(2a_{12}+a_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}-2a_{16}\frac{\partial^4 w}{\partial x\partial y^3}+a_{11}\frac{\partial^4 w}{\partial y^4}=0 \tag{1}$$

for the most general anisotropic elastic medium in the given domain, where

$$\frac{\partial^2 w}{\partial y^2}=\sigma_x,\quad \frac{\partial^2 w}{\partial x^2}=\sigma_y,\quad \frac{\partial^2 w}{\partial x\partial y}=-\tau_{xy}. \tag{2}$$

The coefficients a_{ij}'s are the material constants. It can be shown[4] that in the case, when axes of orthotropy coincide with axes of coordinate system, only a_{11}, a_{12}, a_{21}, a_{26} and a_{66} are nonzero coefficients and $a_{12}=a_{21}$. Under these conditions, the general eqn (1) reduces to

$$a_{22}\frac{\partial^4 w}{\partial x^4}+(2a_{12}+a_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+a_{11}\frac{\partial^4 w}{\partial y^4}=0. \tag{3}$$

A boundary value problem will be formulated for the eqn (3) in half planes P_x and P_y, where

$$P_x=(x, y;\ x>0,\ -\infty<y<\infty)$$
$$P_y=(x, y;\ -\infty<x<\infty,\ y>0).$$

Let the constants a_{11}, a_{12}, a_{21}, a_{22} and a_{66} be given such that:

(i) a_{11}, a_{22} and a_{66} are positive, a_{12} is negative and a_{12} is equal to a_{21};

(ii) $(2a_{12}+a_{66})>2\sqrt{(a_{11}a_{22})}>0$. Let the functions $n_x(s)$ and $t_x(s)$ be given such that;

(iii) $n_x(s)=n_x(-s)$ i.e. symmetrical;

(iv) $t_x(s)=-t_x(-s)$ i.e. antisymmetrical;

(v) $n_x(s)$ and $t_x(s)$ belong to $L_2(0,\infty)$.

The given functions $n_x(s)$ and $t_x(s)$ are called normal and tangential boundary conditions respectively.

Let us denote

(vi)
$$f_x(y)=\int_0^y (y-s)n_x(s)\,\mathrm{d}s \tag{4}$$

(vii)
$$g_x(y)=\int_0^y t_x(s)\,\mathrm{d}s. \tag{5}$$

The boundary value problem for the eqn (3) in half plane P_x is to determine the stress function $w_x(x, y)$ such that the following conditions are satisfied:

(viii) $w_x(x, y)$ is defined in P_x and it has at least four continuous derivatives in P_x with respect to x and y;

(ix) $w_x(x, y)$ satisfies the eqn (3) in P_x.

The boundary conditions should be satisfied in the following sense

(x)
$$\lim_{x\to 0}\int_{-\infty}^{\infty}|w_x(x, y)-f_x(y)|^2\,\mathrm{d}y=0$$

(xi)
$$\lim_{x\to 0}\int_{-\infty}^{\infty}\left|\frac{\partial w_x(x, y)}{\partial x}+g_x(y)\right|^2\mathrm{d}y=0$$

(xii)
$$\lim_{x\to\infty}\int_{-\infty}^{\infty}|w_x(x, y)|^2\,\mathrm{d}y=0$$

(xiii)
$$\lim_{x\to\infty}\int_{-\infty}^{\infty}\left|\frac{\partial w_x(x, y)}{\partial x}\right|^2\mathrm{d}y=0.$$

Positive constants M and N exist such that the following inequality is satisfied in P_x and on the boundary

(xiv) $$\left|\frac{\partial^k w_x(x,y)}{\partial x^i \partial y^j}\right| \le M\,e^{-N|y|}, \quad i+j=k, \quad k=0,1,2,3,4.$$

It can be shown that there exists one solution of the formulated problem. This solution belongs to $L_2(P_x)$. By the term, one solution, we mean that the stress field is unique.

Formally, let us assume that the functions $w_x(x,y)$ belongs to $L_2(-\infty,\infty)$ for arbitrary $x(0<x<\infty)$.

Multiplying the eqn (3) by $e^{-i\alpha y}$ and integrating the products with respect to y over the interval $(-\infty,\infty)$ we obtain an ordinary differential equation for $W_x(x,\alpha)$

$$a_{22}\frac{d^4 W_x}{dx^4}-(2a_{12}+a_{66})\alpha^2\frac{d^2 W_x}{dx^2}+a_{11}\alpha^4 W_x=0 \tag{6}$$

Let

$$a=\sqrt{\left(\frac{(2a_{12}+a_{66})+\surd((2a_{12}+a_{66})-4a_{11}a_{22})}{2a_{22}}\right)}$$
$$b=\sqrt{\left(\frac{(2a_{12}+a_{66})-\surd((2a_{12}+a_{66})-4a_{11}a_{22})}{2a_{22}}\right)} \tag{7}$$

and consequently,

$$a>b>0.$$

The general solution $W_x(x,\alpha)$ of the eqn (6) is given by the following formula

$$W_x(x,\alpha)=c_1\,e^{a|\alpha|x}+c_2\,e^{-a|\alpha|x}+c_3\,e^{b|\alpha|x}+c_4\,e^{-b|\alpha|x}. \tag{8}$$

Using Parseval's equality the conditions (xi)–(xiv) the constants can be evaluated and (8) is simplified as

$$W_x(\chi,\alpha)=\frac{1}{(a-b)}\left[\{a\,e^{-b|\alpha|x}-b\,e^{-a|\alpha|x}\}F_x(\alpha)+\{e^{-a|\alpha|x}-e^{-b|\alpha|x}\}\frac{G_x(\alpha)}{|\alpha|}\right] \tag{9}$$

where $F_x(\alpha)$ and $G_x(\alpha)$ are Fourier–Plancheral transform of $f_x(y)$ and $g_x(y)$ respectively. Since all the conditions for convolution are satisfied, the inverse function $w_x(x,y)$ of $W_x(x,\alpha)$ can be obtained in the following form

$$w_x(x,y)=\frac{ab(a+b)}{\pi}\int_{-\infty}^{\infty}\frac{x^3 f_x(s)\,ds}{\{a^2x^2+(y-s)^2\}\{b^2x^2+(y-s)^2\}}$$
$$+\frac{1}{(a-b)\pi}\int_{-\infty}^{\infty}g_x(s)\ln\sqrt{\left(\frac{b^2x^2+(y-s)^2}{a^2x^2+(y-s)^2}\right)}\,ds. \tag{10}$$

A similar procedure as given above can be followed to get the solution of the problem in half plane P_y. We find the solution has the following form

$$w_y(x,y)=\frac{mn(m+n)}{\pi}\int_{-\infty}^{\infty}\frac{y^3 f_y(s)\,ds}{\{(x-s)^2+m^2y^2\}\{(x-s)^2+n^2y^2\}}$$
$$+\frac{1}{(m-n)\pi}\int_{-\infty}^{\infty}g_y(s)\ln\sqrt{\left(\frac{(x+s)^2+n^2y^2}{(x-s)^2+m^2y^2}\right)}\,ds \tag{11}$$

where

$$f_y(x) = \int_0^x (x-s)n_y(s)\,ds$$
$$g_y(x) = \int_0^x t_y(s)\,ds. \tag{12}$$

The functions $n_y(s)$ and $t_y(s)$ are given normal and tangential boundary conditions respectively,

$$m = \frac{1}{b} \quad \text{and} \quad n = \frac{1}{a}.$$

The stress components are evaluated as

$$\begin{aligned}
\sigma_x^{(x)}(x,y) &= \frac{ab(a+b)}{\pi}\int_{-\infty}^{\infty}\frac{x^3 n_x(s)\,ds}{\{a^2x^2+(y-s)^2\}\{b^2x^2+(y-s)^2\}} \\
&+ \frac{(a+b)}{\pi}\int_{-\infty}^{\infty}\frac{x^2(y-s)t_x(s)\,ds}{\{a^2x^2+(y-s)^2\}\{b^2x^2+(y-s)^2\}} \\
\sigma_y^{(x)}(x,y) &= \frac{ab(a+b)}{\pi}\int_{-\infty}^{\infty}\frac{x(y-s)^2 n_x(s)\,ds}{\{a^2x^2+(y-s)^2\}\{b^2x^2+(y-s)^2\}} \\
&+ \frac{(a+b)}{\pi}\int_{-\infty}^{\infty}\frac{(y-s)^3 t_x(s)\,ds}{\{a^2x^2+(y-s)^2\}\{b^2x^2+(y-s)^2\}}.
\end{aligned} \tag{13}$$

For half plane P_y we have

$$\begin{aligned}
\sigma_x^{(y)}(x,y) &= \frac{mn(m+n)}{\pi}\int_{-\infty}^{\infty}\frac{y(x-s)^2 n_y(s)\,ds}{\{(x-s)^2+m^2y^2\}\{(x-s)^2+n^2y^2\}} \\
&+ \frac{(m+n)}{\pi}\int_{-\infty}^{\infty}\frac{(x-s)^3 t_y(s)\,ds}{\{(x-s)^2+m^2y^2\}\{(x-s)^2+n^2y^2\}} \\
\sigma_y^{(y)}(x,y) &= \frac{mn(m+n)}{\pi}\int_{-\infty}^{\infty}\frac{y^3 n_y(s)\,ds}{\{(x-s)^2+m^2y^2\}\{(x-s)^2+n^2y^2\}} \\
&+ \frac{m+n}{\pi}\int_{-\infty}^{\infty}\frac{x^2(y-s)t_y(s)\,ds}{\{(x-s)^2+m^2y^2\}\{(x-s)^2+n^2y^2\}}.
\end{aligned} \tag{14}$$

3. FORMULATION AND SOLUTIONS OF THE BASIC PROBLEMS FOR ORTHOGONAL WEDGE P_{xy}

To obtain a solution of stress distribution for orthotropic orthogonal wedge loaded by general admissible boundary conditions directly as a whole is difficult. Therefore, we split the problem into basic problems. I and II, which are shown schematically in Figs. 1 and 2. Problem I can further be subdivided into problems A and B and the problem II can be similarly subdivided into C and D as shown in Figs. 1 and 2. In this section, exact theoretical solutions of these basic problems for all admissible boundary conditions will be given. By the term "admissible" we mean the following. If $n(s)$ and $t(s)$ represent normal and tangential boundary conditions respectively, then these conditions are admissible if functions $n(s)$ and $t(s)$ satisfy

$$\int_0^{\infty} |n(s)|^2\,ds < \infty$$
$$\int_0^{\infty} |t(s)|^2\,ds < \infty.$$

It should be noted that Dirac delta function also represents an admissible boundary condition.

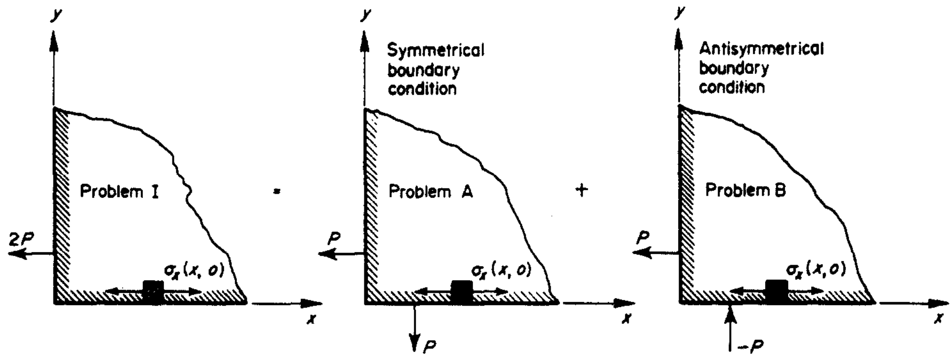

Fig. 1. Problem I. Normal boundary conditions.

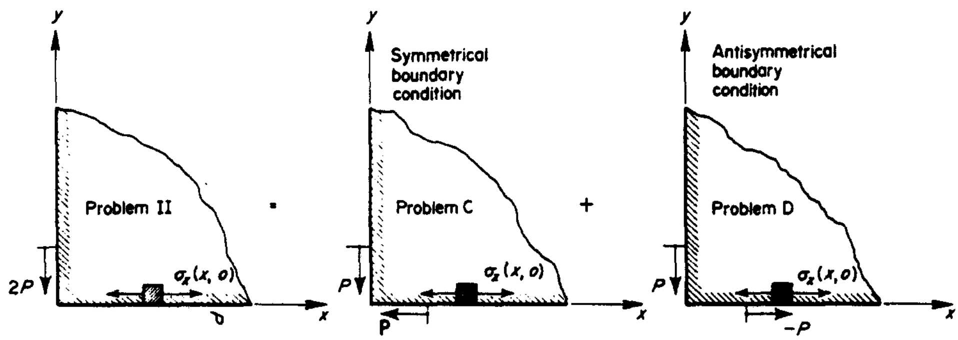

Fig. 2. Problem II. Tangential boundary conditions.

The detailed derivation of the method of solution for the stress distribution for problem A will be given step by step. For problems B, C and D the same can be used.

Let us cut the symmetrically loaded half plane P_x along the axis of symmetry Ox, (Fig. 3) and let us keep the upper portion. Only normal parasitic stress is present along this axis and the distribution of this stress is given by

$$\sigma_y^{(x)} = c_1 \mathrm{K} n_x \tag{15}$$

where

$$\mathrm{K}n_x \equiv \int_0^\infty k_0(x, s) n_x(s)\, \mathrm{d}s, \quad 0 < x < \infty.$$

$$k_0(x, s) = \frac{xs^2}{(a^2x^2 + s^2)(b^2x^2 + s^2)}$$

$$c_1 = \frac{2ab(a+b)}{\pi}, \quad a > b > 0.$$

Let us cut the half plane P_y (Fig. 4) along the axis of symmetry Oy, and let us keep the right quarter plane. Only normal parastic stress is present along this axis and the distribution of this stress is given by

$$\sigma_x^{(y)} = c_2 \mathrm{Q} n_y \tag{16}$$

where

$$\mathrm{Q}n_y \equiv \int_0^\infty q_0(s, y) n_y(s)\, \mathrm{d}s$$

$$q_0(s, y) = \frac{ys^2}{(s^2 + m^2y^2)(s^2 + n^2y^2)}$$

$$c_2 = \frac{2mn(m+n)}{\pi}, \quad m > n > 0.$$

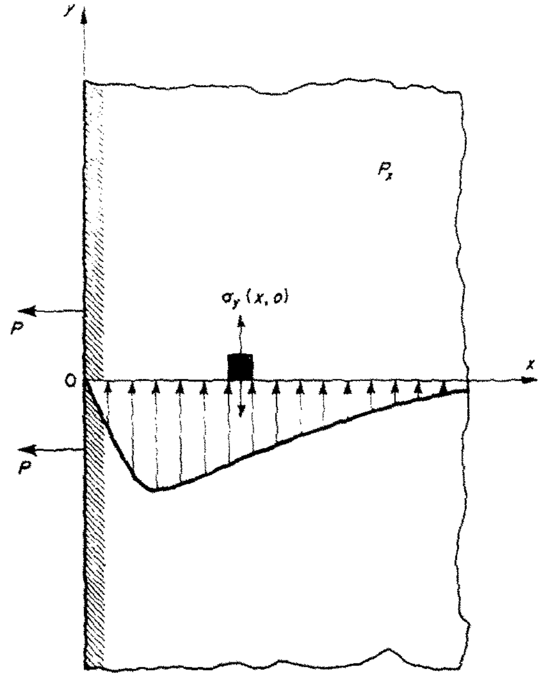

Fig. 3. Half plane P_x subjected to symmetrical normal boundary condition.

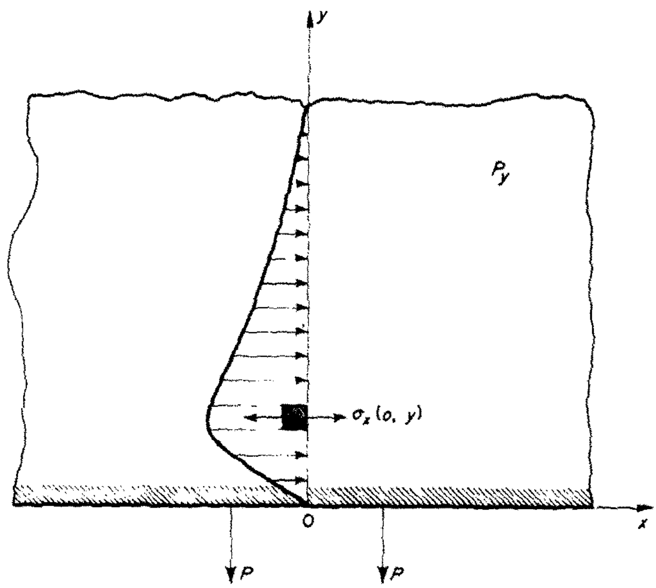

Fig. 4. Half plane P_y subjected to symmetrical normal boundary condition.

Superposing the two wedges with parasitic stresses, we obtain for P_{xy} (Fig. 5)

$$\sigma_n^{(1)} = \begin{cases} n_x + c_1 \mathrm{K} n_x & \text{on } Ox \\ n_y + c_2 \mathrm{Q} n_y & \text{on } Oy. \end{cases} \tag{17}$$

Let us assume that there exist two symmetrical functions $\phi_0^A(y)$ and $\psi_0^A(x)$ from $L_2(0, \infty)$. Let us carry out the same operations as described above for half plane P_x and P_y and we obtain

$$\sigma_n^{(2)} = \begin{cases} \psi_0^A + c_1 \mathrm{K} \phi_0^A & \text{on } Ox \\ \phi_0^A + c_2 \mathrm{Q} \psi_0^A & \text{on } Oy. \end{cases} \tag{18}$$

With these four half planes superposed one over the other, we obtain the following relations

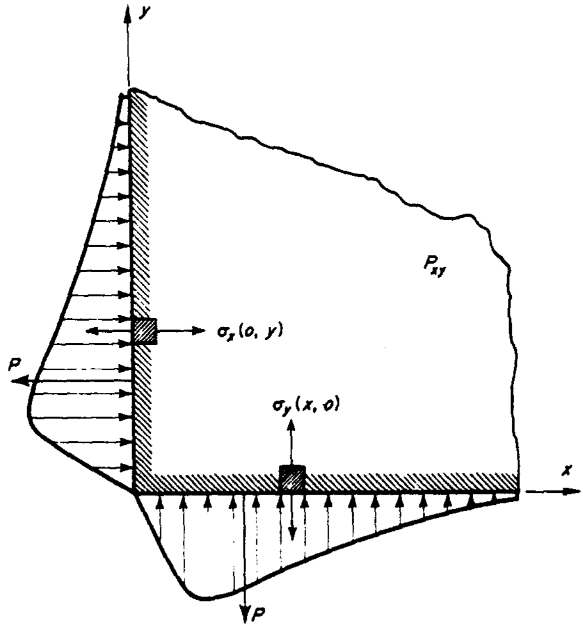

Fig. 5. Quarter plane P_{xy} subjected to normal boundary condition.

on the boundaries of P_{xy}.

$$\sigma_n = \begin{cases} n_x + c_1 K n_x + \psi_0{}^A + c_1 K \phi_0{}^A & \text{on } Ox \\ n_y + c_2 Q n_y + \phi_0{}^A + c_2 Q \psi_0{}^A & \text{on } Oy. \end{cases} \tag{19}$$

These two relations (19) will satisfy the prescribed boundary conditions if and only if the unknown functions $\phi_0{}^A(y)$ and $\psi_0{}^A(x)$ are the solution of following system of integral equations

$$\begin{aligned} \psi_0{}^A + c_1 K \phi_0{}^A &= -c_1 K n_x \quad \text{on} \quad Ox \\ \phi_0{}^A = + c_2 Q \psi_0{}^A &= -Q n_y \quad \text{on} \quad Oy. \end{aligned} \tag{20}$$

Problem A is to determine functions $\phi_0{}^A$ and $\psi_0{}^A$ (from $L_2(0, \infty)$) which are the solution of the system of integral eqns (20). It should be noted that for the system of integral eqns (20) the Fredholm's alternative is not satisfied. Therefore, the usual method of successive approximation cannot be used and a new approach must be followed.

It is shown in the Appendix that the system (20) for the given admissible functions n_x and n_y has a unique solution $\phi_0{}^A$ and $\psi_0{}^A$ which belong to $L_2(0, \infty)$. In a similar manner as given above for the problem A, the relevant problems B, C and D can be formulated and the solutions $\phi_0{}^B$, $\psi_0{}^B$, $\phi_0{}^C$, $\psi_0{}^C$, $\phi_0{}^D$ and $\psi_0{}^D$ can be obtained.

4. GREEN'S FUNCTIONS

In previous section, we have determined the solution of the problem of stress distribution in the wedge for general admissible boundary conditions. It should be noted that for each admissible boundary condition, the same procedure must be repeated, i.e. the system of integral eqns (20) must be solved numerically for each of these different boundary conditions. To avoid this necessity of solving the system of integral equations, we will show a new method to modify the previous procedure. This modification will be shown only for the problem A. It is similar for problems B, C and D.

Let us solve the problem A for special boundary conditions which are prescribed in the following form,

$$n_x(s) = n_y(s) = P\delta(s-1); \quad (P > 0, 0 < s < \infty)$$

where $\delta(s-1)$ is the Dirac function. It is a concentrated force P acting at unit distance from the vertex in positive direction (See Fig. 1).

With these above boundary conditions the eqns (20) become

$$\begin{aligned}\psi_0^A(t)+c_1\int_0^\infty \frac{ts^2\phi_0^A(s)\,ds}{(a^2t^2+s^2)(b^2t^2+s^2)} &= -\frac{c_1 t}{(a^2t^2+1)(b^2t^2+1)}\\ \phi_0^A(t)+c_2\int_0^\infty \frac{ts^2\psi_0^A(s)\,ds}{(a^2t^2+s^2)(b^2t^2+s^2)} &= -\frac{c_2 t}{(a^2t^2+1)(b^2t^2+1)}.\end{aligned} \tag{21}$$

The solution $\psi_0^A(t)$ and $\phi_0^A(t)$ of the system of integral eqns (21) is called Green's function for the problem A.

Let us solve the problem A for the boundary conditions,

$$n_x(s)=n_y(s)=P\delta(s-s_k);\quad P>0,\quad s_k\neq 1$$

where $P\delta(s-s_k)$ is a concentrated force P acting at a distance s_k from the vertex.

The eqns (20) become

$$\begin{aligned}\psi_0^A(t)+c_1\int_0^\infty \frac{ts^2\phi_0^A(s)\,ds}{(a^2t^2+s^2)(b^2t^2+s^2)} &= -\frac{c_1 t s_k^2}{(a^2t^2+s_k^2)(b^2t^2+s_k^2)}\\ \phi_0^A(t)+c_2\int_0^\infty \frac{ts^2\psi_0^A(s)\,ds}{(a^2t^2+s^2)(b^2t^2+s^2)} &= -\frac{c_2 t s_k^2}{(a^2t^2+s_k^2)(b^2t^2+s_k^2)}.\end{aligned} \tag{22}$$

Introducing new variables

$$\alpha=\frac{t}{s_k},\quad 0<\alpha<\infty$$

$$\beta=\frac{s}{s_k},\quad 0<\beta<\infty$$

the eqns (22) after this substitution become

$$\begin{aligned}s_k\psi_0^A(\alpha,s_k)+c_1\int_0^\infty \frac{\alpha\beta^2\phi_0^A(\beta,s_k)\,d\beta}{(a^2\alpha^2+\beta^2)(b^2\alpha^2+\beta^2)} &= -\frac{c_1\alpha}{(a^2\alpha^2+\beta^2)(b^2\alpha^2+\beta^2)}\\ s_k\phi_0^A(\alpha,s_k)+c_2\int_0^\infty \frac{\alpha\beta^2\psi_0^A(\beta,s_k)\,d\beta}{(a^2\alpha^2+\beta^2)(b^2\alpha^2+\beta^2)} &= -\frac{c_2\alpha}{(a^2\alpha^2+\beta^2)(b^2\alpha^2+\beta^2)}.\end{aligned} \tag{23}$$

Comparing (21) and (23) we observe that the functions

$$\begin{aligned}\psi_{0k}^A(t) &= \frac{1}{s_k}\psi_0^A\left(\frac{t}{s_k}\right)\\ \phi_{0k}^A(t) &= \frac{1}{s_k}\phi_0^A\left(\frac{t}{s_k}\right)\end{aligned} \tag{24}$$

where $\psi_{0k}^A(t)$ and $\phi_{0k}^A(t)$ are the solution of the system of integrals eqns (23) for the problem A with symmetrical normal boundary conditions. It must be noted that (24) gives a relationship between the Green's functions and the new functions for which boundary conditions are prescribed. Once the Green's functions $\psi_0^A(t)$ and $\phi_0^A(t)$ are calculated numerically then the functions $\psi_{0k}^A(t)$ and $\phi_{0k}^A(t)$ can be evaluated by simple division. When the functions $\psi_{0k}^A(t)$ and $\phi_{0k}^A(t)$ are known then the stress at any point in P_{xy} for the problem A with the prescribed boundary conditions can be calculated.

In the case of any general admissible symmetrical normal boundary conditions $n_x(s)=n_y(s)=h(s)$ from $L_2(0,\infty)$ for the problem A, relationships similar to (24) can be obtained.

These relationships are given by

$$\psi_{0*}^{A}(t)=\int_0^\infty \psi_0{}^A\left(\frac{t}{s}\right)n_x(s)\frac{\mathrm{d}s}{s}$$

$$\phi_{0*}^{A}(t)=\int_0^\infty \phi_0{}^A\left(\frac{t}{s}\right)n_y(s)\frac{\mathrm{d}s}{s};\quad 0<t<\infty \tag{25}$$

where $\psi_{0*}^{A}(t)$ and ϕ_{0*}^{A} are the solution of the system of integral eqns (20) with any arbitrary admissible boundary conditions. When these functions are known the stresses at any point in P_{xy} can be evaluated.

5. DISCUSSION OF THE RESULTS AND CONCLUSIONS

An exact theoretical solution for the stress distribution in the vicinity of the corner of an orthotropic wedge subjected to general admissible boundary conditions from the Lebesque space $L_2(-\infty,\infty)$, the definition of which is given in Appendix-A, has been presented here. First, the general problem was divided into two main parts, (i) wedge subjected to normal boundary conditions and (ii) wedge subjected to tangential boundary conditions. The Green's function for parts (i) and (ii) are obtained as a solution of a special system of integral eqns (20). In solving this system of integral equations the method of Fredholm or successive approximation could not be used since the Fredholm's alternative (see Appendix B) is not satisfied for this system. So a special method described in Appendix B has been used. Basically, Fourier–Plancherel integral transform technique (refer to Appendix A for definition) has been made use of in solving the system. The Fourier–Plancherel transforms of the kernel functions of this system of integral equations are evaluated using the well-known theory of residues. The detailed derivation of the transforms of kernel functions for both normal and tangential boundary conditions, is shown in Appendix C.

In the numerical evaluation of Green's functions, i.e. eqn B19, improper integrals with range of integration $(0,\infty)$ are encountered. This range $(0,\infty)$ is divided into two parts $(0,\lambda)$ and (λ,∞), where λ is a suitably chosen quantity. λ is chosen such that the integral over the second range (λ,∞), is ngligible. In these calculations λ is chosen as 12.0. For the range $(0,\lambda)$ ordinary quadrature formulae such as Simpson's could not be used because the integrands of these integrals are rapidly oscillating functions. Therefore, numerical integrations have to be performed using a modified form of Filon's method[6]. All these Green's functions thus evaluated for problems A, B, C and D for symmertical boundary conditions are shown in Figs. 6–9 respectively. With the aid of these Green's functions, formulae (eqn 25) for more generalized problem of wedge with arbitrary boundary conditions are developed. Using these formulae the

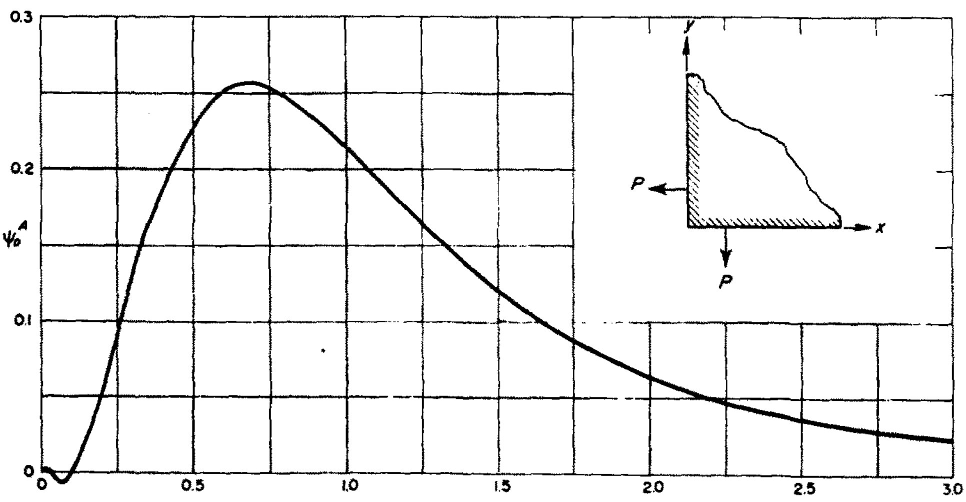

Fig. 6. Green's function $\psi_0{}^A$ for half plane P_y (symmetrical normal boundary condition).

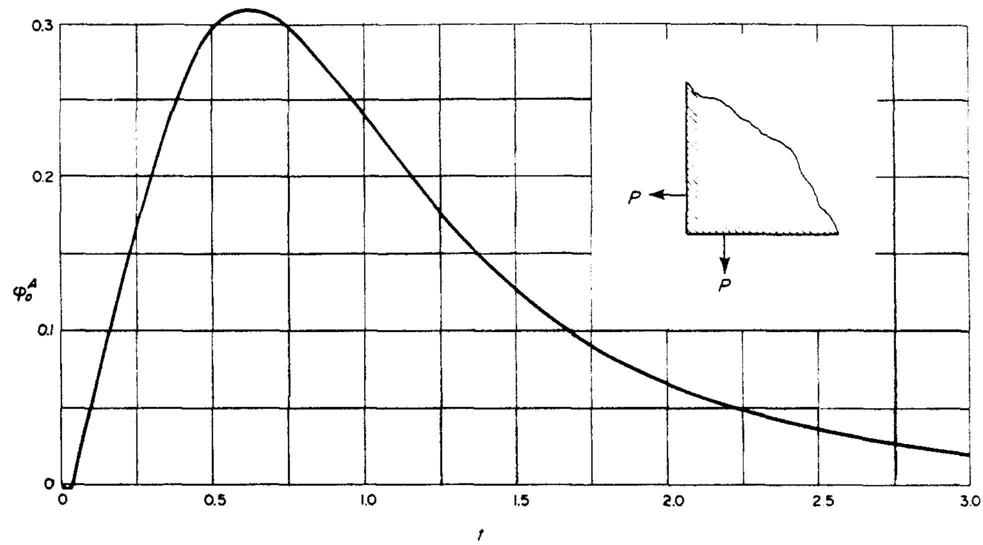

Fig. 7. Green's function $\varphi_0{}^A$ for half plane P_x (symmetrical normal boundary condition).

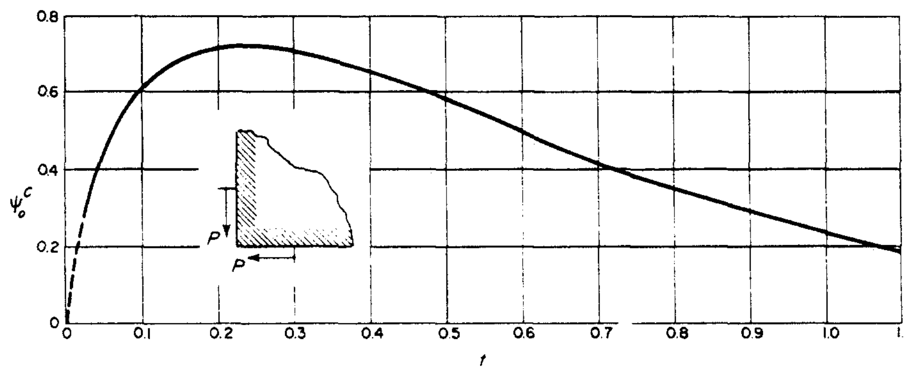

Fig. 8. Green's function $\psi_0{}^C$ for half plane P_y (symmetrical tangential boundary condition).

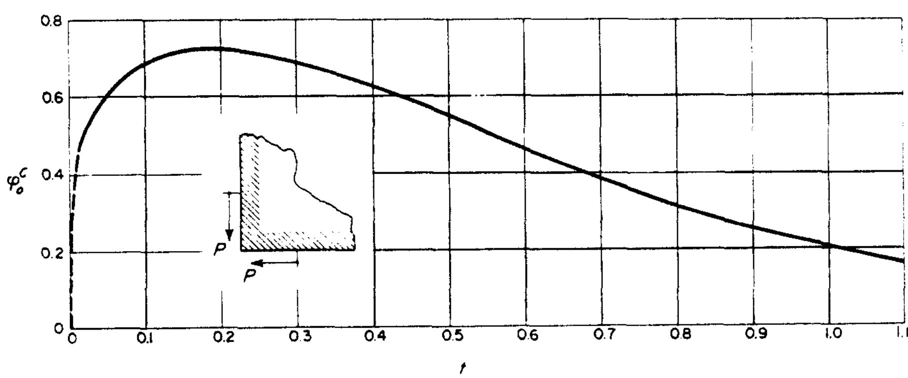

Fig. 9. Green's function $\varphi_0{}^C$ for half plane P_x (symmetrical tangential boundary condition).

stress at any point in the wedge for all admissible arbitrary boundary conditions is very easily calculated. These formulae are of much use to design engineers in analysing wedge problems. These formulae enable them to calculate the stresses without really going through entire analysis given here for each different kind of boundary conditions.

This whole procedure of calculating the stress distribution in the wedge is illustrated by the following example. In the orthogonal wedge which is loaded by a concentrated load acting at a unit distance from the vertex on one face and the other face free from external loading, the stress distribution in the vicinity of the vertex (Fig. 1) is determined. For the wedge with elastic constants $a > b$ the boundary stress along the free face is calculated and the results are shown

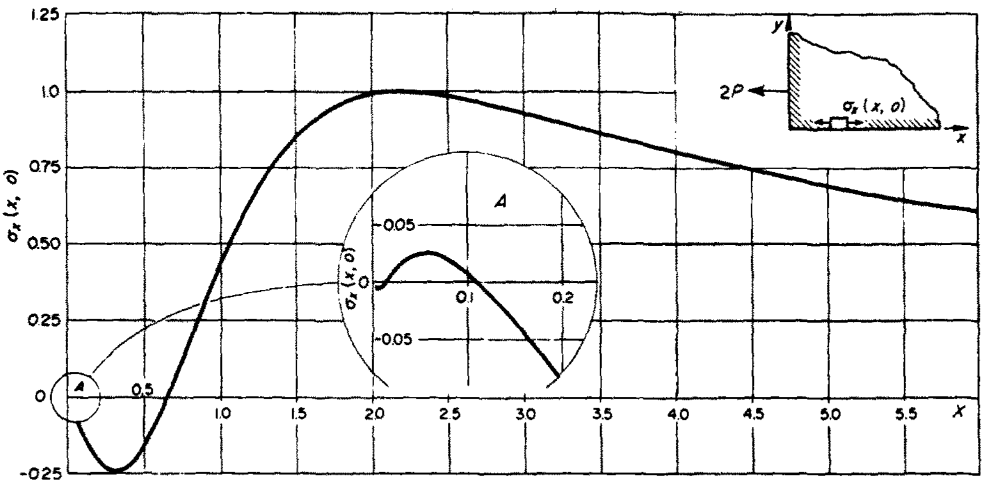

Fig. 10. Stress along the edge in a quarter plane.

in Fig. 10. It can be seen (from the relevant Green's functions) that the stress along the free face near the vertex changes its sign many times. The magnitude of this stress in first reversal of sign is not small. Its value is about 12% of the applied load on the free face. This may be of considerable importance in certain applications.

The method of approach used in this investigation and the solution obtained for the basic problems are fundamental, since a large variety of problems in orthotropic elastic wedge can be solved in a much simpler way by combining these basic solutions and performing a simple integration. In addition, this approach is not limited by the arbitrariness and position of application of boundary loads except these boundary loads must be admissible.

This method can be effectively modified for the analysis of stress distribution in an infinite wedge of any angle ω, providing $0 < \omega \le \pi$. A similar approach can be used in three dimensional problems. Thermal problems for the above sector domains can also be investigated using foregoing procedure.

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APPENDIX A

Fourier–Plancherel transforms

The theory of Fourier–Plancherel transform is well known as providing techniques for solving boundary value problems. The important theorems, which are used to transform a given problem into a relatively simple one, are summarized below.

We say that $f(x)$ belongs to $L_2(-\infty, \infty)$ if $f(x)$ is measurable and

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty. \tag{A1}$$

By

$$\lim_{w \to \infty} \int_{-w}^{w} f(x, \alpha) \, dx$$

we denote a function $F(\alpha)$ such that

$$\lim_{w \to \infty} \int_{-\infty}^{\infty} |F(\alpha)| - \int_{-W}^{W} |f(x, \alpha) \, dx|^2 \, d\alpha = 0 \tag{A2}$$

Let $\phi(x)$ be a real function such that $\int_{-\infty}^{\infty} |\phi(x)|^2\, dx < \infty$. Consequently, a function $\Phi_w(\alpha)$ exists in $L_2(-\infty, \infty)$ such that

$$\lim_{w\to\infty} \int_{-\infty}^{\infty} |\Phi_w(\alpha) - \Phi(\alpha)|^2\, d\alpha = 0 \tag{A3}$$

where

$$\Phi_w(\alpha) = \int_{-w}^{w} \phi(x)\, e^{-i\alpha x}\, dx. \tag{A4}$$

This limit is called Fourier–Plancherel transform of $\phi(x)$ in $L_2(-\infty, \infty)$.
An inversion formula exists in $L_2(-\infty, \infty)$ such that

$$\lim_{w\to\infty} \int_{-\infty}^{\infty} |\phi(x) - \phi_w(x)|^2\, dx = 0 \tag{A5}$$

where

$$\phi_w(x) = \frac{1}{2\pi} \int_{-w}^{w} \Phi(\alpha)\, e^{i\alpha x}\, d\alpha. \tag{A6}$$

The limit (A5) can also be written in the following form

$$\phi(x) = \lim_{w\to\infty} \frac{1}{2\pi} \int_{-w}^{w} \Phi(\alpha)\, e^{-i\alpha x}\, d\alpha. \tag{A7}$$

The Parseval's equality in the class $L_2(-\infty, \infty)$ is given by

$$\int_{-\infty}^{\infty} |\phi(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi(\alpha)|^2\, d\alpha \tag{A8}$$

Let $h(x)$ belong to $L_2(-\infty, \infty)$ and $L_1(-\infty, \infty)$. In addition, let $\phi(x)$ belong to $L_2(-\infty, \infty)$; then the function

$$f(x) = \int_{-\infty}^{\infty} h(x - s)\phi(s)\, ds \tag{A9}$$

belongs to $L_2(-\infty, \infty)$ and

$$F(\alpha) = H(\alpha)\phi(\alpha). \tag{A10}$$

The function $f(x)$ is called convolution of $h(x)$ and $\phi(x)$.
Note that the original functions are represented by the small letters and the respective Fourier–Plancherel transforms are given by the capital letters.

APPENDIX B

Solution of special system of integral equations

The solutions of basic problems A, B, C and D depend on the behaviour of the following system of integral equations with two unknown functions ψ_0 and ϕ_0

$$\psi_0(t) + \int_0^{\infty} k_0(t, s)\phi_0(s)\, ds = f_0(t),$$

$$\phi_0(t) + \int_0^{\infty} q_0(t, s)\psi_0(s)\, ds = h_0(t), \quad 0 < t < \infty \tag{B1}$$

where, for normal boundary conditions, the Kernel functions have the following form

$$k_0(t, s) = \frac{2ab(a+b)}{\pi} \frac{ts^2}{(a^2t^2 + s^2)(b^2t^2 + s^2)}; \quad a > b > 0 \tag{B2}$$

$$q_0(t, s) = \frac{2mn(m+n)}{\pi} \frac{ts^2}{(m^2t^2 + s^2)(n^2t^2 + s^2)} \quad a > b > 0 \tag{B3}$$

$f_0(t)$ and $h_0(t)$ are given functions from $L_2(-\infty, \infty)$.
Since

$$\int_0^{\infty} \int_0^{\infty} |k_0(t, s)|^2\, dt\, ds = \infty$$

$$\int_0^{\infty} \int_0^{\infty} |q_0(t, s)|^2\, dt\, ds = \infty$$

the Fredholm's alternative is not satisfied.

First of all, we must prove that the operator

$$K_0 g_0 \equiv \int_0^\infty k_0(t, s) g_0(s)\, ds \tag{B4}$$

transforms $L_2(0, \infty)$ into itself, i.e. we must prove that $K_0 g_0$ belongs to $L_2(0, \infty)$ for $g_0(s) \in L_2(0, \infty)$.

Using the facts that

$$\int_0^\infty k_0(t, s)\, ds = c_1 < \infty$$

$$\int_0^\infty k_0(t, s)\, dt = c_2 < \infty$$

and Schwarz–Bunyakovskii inequality, it follows that

$$\left[\int_0^\infty k_0(t, s) g_0(s)\, ds\right]^2 \le c_1 \int_0^\infty k_0(t, s) g_0^2(s)\, ds \tag{B5}$$

After integrating the inequality (B5) with respect to t over the interval $(0, \infty)$ we obtain

$$\int_0^\infty \left[\int_0^\infty k_0(t, s) g_0(s)\, ds\right]^2 dt \le c_1 \int_0^\infty \int_0^\infty k_0(t, s) g_0^2(s)\, ds\, dt$$

$$\le c_1 \int_0^\infty g_0^2(s) \int_0^\infty k_0(t, s)\, dt\, ds \le c_1 c_2 \int_0^\infty g_0^2(s)\, ds. \tag{B6}$$

Therefore, the last inequality becomes

$$\int_0^\infty (K_0 g_0)^2\, ds \le c_1 c_2 \int_0^\infty g_0^2(s)\, ds \tag{B7}$$

and since $g_0(s)$ belongs to $L_2(0, \infty)$ it is seen that the operator (B4) transforms $L_2(0, \infty)$ into itself; this completes the proof.

Note

For tangential boundary conditions the kernel functions have the following form:

$$k_0^*(t, s) = \frac{2(a+b)}{\pi} \frac{s^3}{(a^2t^2 + s^2)(b^2t^2 + s^2)}; \quad 0 < t, s < \infty \tag{B8}$$

$$q_0^*(t, s) = \frac{2(m+n)}{\pi} \frac{s^3}{(m^2t^2 + s^2)(n^2t^2 + s^2)}; \; m > n > 0. \tag{B9}$$

For this case, a method similar to that given above can be followed to prove that the operator (B4) for $m > n > 0$ transforms $L_2(0, \infty)$ into itself.

Method of solution

Before solving the system of integral eqns (B1) we will transform this system into a convenient and suitable form.

Let

$$t = e^v, \quad s = e^u, \tag{B10}$$

then the following relation is valid

$$\int_0^\infty \phi_0^2(t)\, dt = \int_{-\infty}^\infty \phi_0^2(e^v)\, e^v\, dv.$$

Denoting

$$\phi(v) = \phi_0(e^v)\, e^{v/2} \tag{B11}$$

we find that function $\phi(v)$ belongs to $L_2(0, \infty)$. Similarly, it can be shown that the function ψ, f and h belong to $L_2(0, \infty)$.

By substituting ϕ, ψ, f and h into (B1) we finally obtain the following system of integral equations

$$\psi(v) + \int_{-\infty}^\infty k(v-u)\phi(u)\, du = f(v),$$

$$\phi(v) + \int_{-\infty}^\infty q(v-u)\psi(u)\, du = h(v) \tag{B12}$$

where

$$k(v-u) = \frac{e^{(3/2)(v-u)}}{(1 + a^2 e^{2(v-u)})(1 + b^2 e^{2(v-u)})}$$

$$q(v-u) = \frac{e^{(3/2)(v-u)}}{(1 + m^2 e^{2(v-u)})(1 + n^2 e^{2(v-u)})}. \tag{B13}$$

Since both the functions ψ and ϕ and the kernel functions (B13) satisfy the conditions of convolution theorem, the Fourier–Plancherel transform theory can be used. By transformation, the system of integral eqns (B12) become

$$\Psi(\alpha)+K(\alpha)\Phi(\alpha)=F(\alpha)$$
$$\Phi(\alpha)+Q(\alpha)\Psi(\alpha)=H(\alpha). \tag{B14}$$

The necessary and sufficient conditions for the existence of unique solution of system (B14) is satisfied, i.e.

$$\Delta(\alpha)=\begin{vmatrix}1 & K(\alpha)\\ Q(\alpha) & 1\end{vmatrix}=1-K(\alpha)Q(\alpha)\neq 0 \quad \text{for all} \quad \alpha\epsilon(-\infty,\infty). \tag{B15}$$

Indeed, it can be proved that

$$|K(\alpha)|\leq|K(0)|=\frac{\sqrt{(2)}ab}{\sqrt{(a)}+\sqrt{(b)}}$$
$$|Q(\alpha)|\leq|Q(0)|=\frac{\sqrt{(2)}mn}{\sqrt{(m)}+\sqrt{(n)}} \tag{B16}$$
$$|K(\alpha)Q(\alpha)|<K(0)Q(0)=\frac{\sqrt{(2)}ab}{\sqrt{(a)}+\sqrt{(b)}}\frac{\sqrt{(2)}mn}{\sqrt{(m)}+\sqrt{(n)}}=D_0.$$

In this special case when $m=(1/b)$ and $n=(1/a)$ we have

$$D_0=\frac{a\sqrt{(ab)}}{(a+b)+2\sqrt{(ab)}}<1 \quad \text{for} \quad a>b>0. \tag{B17}$$

Therefore, $\Delta(\alpha)\neq 0$ for all values of α.

We finally obtain the solution of the system (B14)

$$\Psi(\alpha)=\frac{F(\alpha)-K(\alpha)H(\alpha)}{1-K(\alpha)Q(\alpha)}$$
$$\Phi(\alpha)=\frac{H(\alpha)-Q(\alpha)F(\alpha)}{1-K(\alpha)Q(\alpha)}. \tag{B18}$$

It is to be noted that $\psi(\alpha)$ and $\phi(\alpha)$ given by (B18) belong to $L_2(-\infty,\infty)$ as well as $L_1(-\infty,\infty)$. Therefore, the inverse of these functions are given by the following

$$\psi_0(t)=\frac{1}{\pi\sqrt{(t)}}\int_0^\infty[\Psi_1(\alpha)\cos(\alpha\ln t)-\Psi_2(\alpha)\sin(\alpha\ln t)]\,d\alpha$$
$$\phi_0(t)=\frac{1}{\pi\sqrt{(t)}}\int_0^\infty[\Phi_1(\alpha)\cos(\alpha\ln t)-\Phi_2(\alpha)\sin(\alpha\ln t)]\,d\alpha \tag{B19}$$

where $\Psi_1(\alpha)$ and $\Phi_1(\alpha)$ are the real parts of $\Psi(\alpha)$ and $\phi(\alpha)$ $\Phi(\alpha)$ respectively and $\Psi_2(\alpha)$ and $\Phi_2(\alpha)$ are the imaginary parts of $\Psi(\alpha)$ and $\Phi(\alpha)$ respectively.

APPENDIX C

Evaluation of Fourier–Plancherel transform of the functions k(v) *and* q(v)

In Appendix B the solution of special system of integral equations was derived. This solution for all admissible boundary conditions depends strongly on the Fourier–Plancherel transforms of the kernel functions $k(v)$ and $q(v)$ given by (B14). In this section, the Fourier–Plancherel transforms of these functions will be evaluated and is summarized in the following theorem.

Theorem:

Let the constants a and b and the function $k(v)$ be given such that

(i) $$a>b>0$$

(ii) $$k(v)=\frac{e^{3/2v}}{(1+a^2e^{2v})(1+b^2e^{2v})}$$

then the Fourier–Plancherel transform $K(\alpha)$ of $K(v)$ in $L_2(-\infty,\infty)$ is

$$K(\alpha)\equiv K_1(\alpha)+iK_2(\alpha)=\int_{-\infty}^{\infty}k(v)\,e^{-i\alpha v}\,dv; \tag{C1}$$

where

$$K_1(\alpha)=\frac{\pi}{\sqrt{(2)}(a^2-b^2)}[H_1(\alpha)G_2(\alpha)-H_2(\alpha)G_1(\alpha)]$$
$$K_2(\alpha)=-\frac{\pi}{\sqrt{(2)}(a^2-b^2)}[H_1(\alpha)G_1(\alpha)+H_2(\alpha)G_2(\alpha)] \tag{C2}$$

$$H_1(\alpha)=\frac{ch\frac{\pi}{2}\alpha}{ch\pi\alpha}$$

$$H_2(\alpha)=\frac{ch\frac{\pi}{2}\alpha}{ch\pi\alpha} \tag{C3}$$

$$G_1(\alpha)=\sqrt{(a)}\sin(x_1\alpha)-\sqrt{(b)}\sin(x_2\alpha)$$

$$G_2(\alpha)=\sqrt{(a)}\cos(x_1\alpha)-\sqrt{(b)}\cos(x_2\alpha) \tag{C4}$$

$$x_1=\ln\left(\frac{1}{a}\right)$$

$$x_2=\ln\left(\frac{1}{b}\right) \tag{C5}$$

Proof:
Consider the integral

$$\int_{c+} f(z)\,\mathrm{d}z=\int_{c^+}\frac{e^{3/2z}\,e^{-i\alpha z}}{(1+a\,e^{2z})(1+b\,e^{2z})}\mathrm{d}z \tag{C6}$$

where c^+ is a rectangle having vertices at $-R$, R, $R+\pi i$ and $-R+\pi i$ and $R>\max(|x_1|,|x_2|)$
The only poles of the integrand $f(z)$ enclosed by the rectangle c^+ are

$$Z_1^+=x_1+i\frac{\pi}{2}$$

$$Z_2^+=x_2+i\frac{\pi}{2} \tag{C7}$$

where

$$x_1=\ln\left(\frac{1}{a}\right),\quad x_2=\ln\left(\frac{1}{b}\right).$$

The residue of $f(z)$ at z_1^+ is

$$\lim_{z\to z_1^+}(z-z_1^+)f(z)=-\frac{a^2\,e^{\omega z_1^+}}{2(a^2-b^2)},\quad \omega=\frac{3}{2}-i\alpha \tag{C8}$$

Similarly, the residue of $f(z)$ at z_2^+ is

$$\lim_{z\to z_2^+}(z-z_2^+)f(z)=\frac{b^2\,e^{\omega z_2^+}}{2(a^2-b^2)},\quad \omega=\frac{3}{2}-i\alpha. \tag{C9}$$

By residue theorem,

$$\oint_{c^+} f(z)\,\mathrm{d}z=\frac{\pi i}{(a^2-b^2)}[b^2\,e^{\omega z_2^+}-a^2\,e^{\omega z_1^+}] \tag{C10}$$

The l.h.s. of (C10) can be written as

$$\oint_{c^+} f(z)\,\mathrm{d}z=\int_{-R}^{R} f(x)\,\mathrm{d}x+\int_0^{\pi} f(R+iy)i\,\mathrm{d}y+\int_R^{-R} f(x+i\pi)\,\mathrm{d}x+\int_{\pi}^{0} f(-R+iy)i\,\mathrm{d}y. \tag{C11}$$

As $R\to\infty$, the second and fourth integrals in (C11) approach zero for $\alpha\le 0$. To prove this, let us consider, for example, the second integral of (C11)

$$\left|\int_0^{\pi} f(R+iy)i\,\mathrm{d}y\right|\le\frac{e^{-(5/2)R}}{(a^2-e^{-2R})(b^2-e^{-2R})}\frac{e^{\pi\alpha}-1}{\alpha},\quad \alpha\le 0 \tag{C12}$$

and the result follows since the integral for $\alpha\le 0$ approaches zero as R tends to infinity. In a similar manner we can show that the fourth integral of (C11) for $\alpha\le 0$ approaches zero as $R\to\infty$.
Since

$$f(x+i\pi)=-e^{i\omega\pi}f(x)$$

and

$$f(x)=k(x)\,e^{-i\alpha x}.$$

We obtain,

$$K^+(\alpha) = K_1^+(\alpha) + iK_2(\alpha)$$
$$= \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \frac{\pi i}{(1 - e^{i\omega\pi})(a^2 - b^2)}(b^2 e^{\omega z_2^+} - a^2 e^{\omega z_1^+}) \tag{C13}$$

After separating real and imaginary parts of $K^+(\alpha)$, the following results for $\alpha \le 0$ are obtained.

$$\begin{aligned} K_1^+(\alpha) &= \frac{\pi}{\sqrt{(2)}(a^2 - b^2)}[H_1(\alpha)G_2(\alpha) - H_2(\alpha)G_1(\alpha)] \\ K_2^+(\alpha) &= -\frac{\pi}{\sqrt{(2)}(a^2 - b^2)}[H_1(\alpha)G_1(\alpha) + H_2(\alpha)G_2(\alpha)]. \end{aligned} \tag{C14}$$

In a similar manner, using the rectangle c^- in the lower half plane, the corresponding result $K^-(\alpha)$ for $\alpha \ge 0$ can be obtained.

These two results $K^+(\alpha)$ and $K^-(\alpha)$ can be combined and the final result can be written in the form

$$\begin{aligned} K(\alpha) &\equiv K_1(\alpha) + iK_2(\alpha) \\ K_1(\alpha) &= \frac{\pi}{\sqrt{(2)}(a^2 - b^2)}[H_1(\alpha)G_2(\alpha) - H_2(\alpha)G_1(\alpha)] \\ K_2(\alpha) &= -\frac{\pi}{\sqrt{(2)}(a^2 - b^2)}[H_1(\alpha)G_1(\alpha) + H_2(\alpha)G_2(\alpha)]. \end{aligned} \tag{C15}$$

For tangential boundary conditions, the Fourier-Plancherel transform of the following functions have to be evaluated.

$$k^*(v) = \frac{e^{(1/2)v}}{(1 + a^2 e^{2v})(1 + b^2 e^{2v})}, \quad a > b > 0. \tag{C16}$$

It can be proved that the Fourier-Plancherel transform of $k^*(v)$ is

$$K^*(\alpha) \equiv K_1^*(\alpha) + K_2^*(\alpha) \tag{C17}$$

where

$$\begin{aligned} K_1^*(\alpha) &= -\frac{\pi}{\bar{2}(a^2 - b^2)}[H_2(\alpha)G_1^*(\alpha) - H_1(\alpha)G_2^*(\alpha)] \\ K_2^*(\alpha) &= -\frac{\pi}{\sqrt{(2)}(a^2 - b^2)}[H_2(\alpha)G_2^*(\alpha) - H_1(\alpha)G_1^*(\alpha)] \end{aligned} \tag{C18}$$

$$\begin{aligned} G_1^*(\alpha) &= a\sqrt{(a)}\sin(x_1\alpha) - b\sqrt{(b)}\sin(x_2\alpha) \\ G_2^*(\alpha) &= a\sqrt{(a)}\cos(x_1\alpha) - b\sqrt{(b)}\sin(x_2\alpha). \end{aligned} \tag{C19}$$

$H_1(\alpha)$ and $H_2(\alpha)$ are given by (C3) and x_1, x_2 are given by (C5).

The same method is used to determine the Fourier-Plancherel transforms of the functions $q(v)$ and $q^*(v)$ and since this is similar to the procedure given above, it will not be discussed.